数理解析研究所講究録
第 1836 巻 2013 年 181-191

NOTE ON THE ASYMPTOTICS OF THE HIGHER DIMENSIONAL REIDEMEISTER TORSION FOR BRIESKORN MANIFOLDS

YOSHIKAZU YAMAGUCHI

1. Introduction

In this note, we discuss the asymptotics of the higher dimensional Reidemeister torsion for Brieskorn manifolds $\Sigma(p,q,npq+1)$ through explicit computations. Our computation is based on a cut and paste method to construct a Brieskorn manifold $\Sigma(p,q,npq+1)$. We will see that the Reidemeister torsion for $\Sigma(p,q,npq+1)$ is expressed as a product of Reidemeister torsions for circles. This computation result allows us to describe the properties of the Reidemeister torsion for $\Sigma(p,q,npq+1)$ by observing the Reidemeister torsion for the circle. As results, we can see that the asymptotic behavior of the higher dimensional Reidemeister torsion for a Brieskorn manifold and determine the limit of the leading coefficient. This is the purpose of this note. Our results can be extended to more general situation. We refer to [11] for an extension to orientable closed Seifert 3-manifolds. The observations of this note are based on the results in [10, 11] by the author.

We will show the following explicit form of the higher dimensional Reidemeister torsion for a Brieskorn manifold $\Sigma(p,q,npq+1)$ and an $\mathrm{SL}_2(\mathbb{C})$-representation ρ. The higher dimensional Reidemeister torsion is defined by the induced n-dimensional representation ρ_n from ρ. We denote by $\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_n)$ this Reidemeister torsion.

Main theorem 1 (Theorem 3.3). *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(\Sigma(p,q,npq+1))$ and r be $|npq+1|$. Suppose that ρ is irreducible and satisfies the acyclicity condition. Then the higher dimensional Reidemeister torsion $\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})$ is expressed as*

$$\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N}) = \frac{2^{2N}}{\prod_{k=1}^{N}\left\{4\sin^2\frac{(2k-1)a\pi}{2p}\cdot 4\sin^2\frac{(2k-1)b\pi}{2q}\cdot 4\sin^2\frac{(2k-1)(cpq-r)\pi}{2r}\right\}}$$

where a, b and c are integers determined by the $\mathrm{SL}_2(\mathbb{C})$-representation ρ.

We are interested in the asymptotic behavior of $\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|$ on $N\geq 1$. We will compute the limit of the leading coefficient and show that the maximal value is given by $-\chi\log 2$ where χ is the Euler characteristic of the base orbifold of $\Sigma(p,q,npq+1)$ as a Seifert manifold.

Main theorem 2 (Theorem 3.7 and Corollary 3.9). *Suppose that ρ is irreducible and satisfies the acyclicity condition. Then we can describe the asymptotic behavior as follows:*

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|}{(2N)^2} = 0$$
$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|}{2N} = \left(1-\frac{1}{p'}-\frac{1}{q'}-\frac{1}{r'}\right)\log 2$$

Received December 31, 2012.

where $p' = \frac{p}{(p,a)}$, $q = \frac{q}{(q,b)}$ *and* $r = \frac{r}{(r,c)}$. *Here* (p, a) *denotes the g.c.d of* p *and* a.

Moreover there exists an irreducible $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *satisfying that the acyclicity condition and* $(p, a) = (q, b) = (r, c) = 1$. *In particular, such an* $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *gives the maximal value of* $\lim_{N\to\infty} \log|\mathrm{Tor}(\Sigma(p, q, npq + 1); \rho_{2N})|/(2N)$ *by*

$$-\chi \log 2 = \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) \log 2.$$

In the context of the Reidemeister torsion for a Seifert manifold, we carry out our computation under the assumption that an $\mathrm{SL}_{2N}(\mathbb{C})$-representation sends every general fiber to $-I_{2N}$ where I_{2N} denotes the identity matrix in $\mathrm{SL}_{2N}(\mathbb{C})$. For a Seifert manifold, the Reidemeister torsion can be also computed explicitly with more general special linear representations. This can be found in the paper [4] by Teruaki Kitano.

2. Preliminaries

2.1. **Review of higher dimensional Reidemeister torsion.** We review the higher dimensional Reidemeister torsion very briefly. For details on the definition of the Reidemeister torsion, we refer to Turaev's book [9]. We also refer to [6, 10] on the definition of the higher dimensional Reidemeister torsion.

The higher dimensional Reidemeister torsion of a finite CW-complex W is defined as the torsion of the twisted chain complex of W. The twisted chain complex $C_*(W; V)$ is defined as $V \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z})$, by choosing a homomorphism from $\pi_1(W)$ into $\mathrm{GL}(V)$, where $\widetilde{W}$ is the universal cover of W. In the definition of $C_*(W; V)$, the vector space V is a right $\mathbb{Z}[\pi_1(W)]$-module through the representation ρ^{-1}.

Definition 2.1. Let W be a finite CW-complex and ρ an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$.

(1) We denote by ρ_n the composition of ρ with the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.
(2) The n-th higher dimensional Reidemeister torsion $\mathrm{Tor}(W; \rho_n)$ is defined as the torsion of $C_*(W; V_n)$ when $C_*(W; V_n)$ is acyclic (i.e., $H_*(W; V_n) = 0$). Here V_n is the n-dimensional vector space equipped with the action of $\mathrm{SL}_2(\mathbb{C})$.

Remark 2.2. The n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ is given by the vector space V_n of homogeneous polynomials $p(z_1, z_2)$ with degree $n - 1$ and the following action of $\mathrm{SL}_2(\mathbb{C})$:

$$A \cdot p(z_1, z_2) = p(A^{-1} \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)), \quad \forall A \in \mathrm{SL}_2(\mathbb{C}).$$

Actually we need only the explicit form of the higher Reidemeister torsion for the circle. We will see the details on the case of the circle in the next Subsection.

2.2. **Example for the circle.** We begin with the twisted chain complex for S^1 and a $\mathrm{GL}(V)$-representation of $\pi_1(S^1)$. We denote by γ a generator of $\pi_1(S^1)$. When we think of S^1 as the union $e^0 \cup e^1$, the twisted chain complex $C_*(S^1; V)$ is expressed as follows:

$$0 \to C_1(S^1; V)(\simeq V) \xrightarrow{\partial_1} C_0(S^1; V)(\simeq V) \to 0$$
$$\boldsymbol{v} \otimes \tilde{e}^1 \mapsto \boldsymbol{v} \cdot \gamma \otimes \tilde{e}^0 - \boldsymbol{v} \otimes \tilde{e}^0.$$

The boundary operator ∂_1 is given by $(\rho(\gamma)^{-1} - I)$. The Reidemeister torsion for S^1 is determined by this matrix presentation of ∂_1.

Proposition 2.3. *Let ρ be a $\mathrm{GL}(V)$-representation of $\pi_1(S^1)(=\langle\gamma\rangle)$. If $\rho(\gamma)$ does not have the eigenvalue 1, then the twisted chain complex $C_*(S^1;V_2)$ is acyclic, i.e, $H_*(S^1;V)=0$. Moreover the Reidemeister torsion is expressed as*

$$\mathrm{Tor}(S^1;\rho)=\frac{1}{\det(\rho(\gamma)^{-1}-I)}.$$

We are interested in the sequence of ρ_n induced by an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(S^1)$ and the corresponding Reidemeister torsion. When the eigenvalues of $\rho(\gamma)$ are $\zeta^{\pm 1}$, direct calculations show that the eigenvalues of $\rho_n(\gamma)$ are given by

$$\begin{cases}\{\zeta^{\pm(2N-1)},\zeta^{\pm(2N-3)},\dots,\zeta^{\pm 3},\zeta^{\pm 1}\} & \text{if } n=2N\\ \{\zeta^{\pm(2N)},\zeta^{\pm(2N-2)},\dots,\zeta^{\pm 2},1\} & \text{if } n=2N+1.\end{cases}$$

We require the acyclicity of the twisted chain complex for ρ_n. Hence we will focus on even dimensional representations ρ_{2N}. Then the corresponding Reidemeister torsion of S^1 is expressed in terms of the eigenvalues $\zeta^{\pm 1}$ of $\rho(\gamma)$ as follows.

Proposition 2.4. *Suppose that the eigenvalue ζ of $\rho(\gamma)$ is not any $(2j-1)$-th root of unity for all $j=1,\dots N$. Then we can express the Reidemeister torsion $\mathrm{Tor}(S^1;\rho_{2N})$ as*

$$\begin{aligned}\mathrm{Tor}(S^1;\rho_{2N})&=\frac{1}{\det(\rho_{2N}(\gamma)^{-1}-I)}\\ &=\left\{\prod_{k=1}^{N}(\zeta^{2k-1}-1)(\zeta^{-(2k-1)}-1)\right\}^{-1}.\end{aligned}$$

Corollary 2.5. *If $\rho(\gamma)$ has the order of $2p$, then for every $N\geq 1$ the twisted chain complex $C_*(S^1;V_{2N})$ is acyclic and the Reidemeister torsion $\mathrm{Tor}(S^1;\rho_{2N})$ is given by the following product:*

$$\left\{\prod_{k=1}^{N}4\sin^2\frac{(2k-1)a\pi\sqrt{-1}}{2p}\right\}^{-1}$$

where a is an odd integer such that $\zeta=e^{a\pi\sqrt{-1}/p}$.

2.3. **Brieskorn manifold $\Sigma(p,q,npq+1)$.** Every Brieskorn manifold $\Sigma(p,q,npq+1)$ can be obtained by $(1/n)$-surgery along the (p,q)-torus knot K. We have the decomposition of $\Sigma(p,q,npq+1)$ as the union $E_K\cup_{\partial E_K}D^2\times S^1$.

Moreover we can divide every torus knot exterior into the union of two solid tori. This decomposition is expressed as

$$E_K=D^2\times S^1\cup_A S^1\times D^2$$

where A denotes the annulus in the torus on which the (p,q)-torus knot K lies. This decomposition arises the following presentation of $\pi_1(E_K)$:

$$\pi_1(E_K)=\langle x,y\,|\,x^p=y^q\rangle$$

where x denotes the homotopy class of $\{*\}\times S^1$ and y denotes that of $S^1\times\{*\}$.

We denote by z the element $x^p(=y^q)$ in $\pi_1(E_K)$, which is the homotopy class of $S^1\times\{*\}$ in $A=S^1\times[-1,1]$. It is known that z is a generator of the center in $\pi_1(E_K)$, which is an infinite cyclic subgroup.

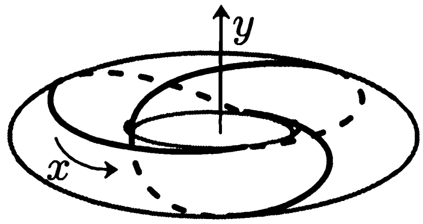

FIGURE 1. Decomposition of the (2, 3)-torus knot exterior

Summarizing the above, we have seen that a Brieskorn manifold $\Sigma(p, q, npq + 1)$ is decomposed as

$$\Sigma(p, q, npq + 1) = (D^2 \times S^1 \cup_A S^1 \times D^2) \cup_{\partial E_K} D^2 \times S^1.$$

The fundamental group of $\Sigma(p, q, npq + 1)$ is expressed as

$$\pi_1(\Sigma(p, q, npq + 1)) = \langle x, y \mid x^p = y^q, m\ell^n = 1\rangle$$

where m and ℓ denote the meridian and the longitude given by the equality that $m = x^{-u}y^v(pv - qu = 1)$ and $\ell = m^{pq}x^{-p}$. Note that ℓ also denotes the homotopy class of the core of $D^2 \times S^1$ glued to E_K in $\pi_1(\Sigma(p, q, npq + 1))$ since the surgery slope is $1/n$.

We will see that the Reidemeister torsion for $\Sigma(p, q, npq + 1)$ is given by the product of those for the circles corresponding to x, y, z and ℓ in $\pi_1(\Sigma(p, q, npq + 1))$.

This is due to that the Reidemeister torsions of a solid torus and an annulus coincide with those of the spines.

3. Higher dimensional Reidemeister torsion for Brieskorn manifolds

3.1. Irreducible $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(E_K)$ and $\pi_1(\Sigma(p, q, npq + 1))$. To describe the Reidemeister torsion for $\Sigma(p, q, npq + 1)$ explicitly, we need to find the eigenvalues of matrices corresponding to x, y, z and ℓ in $\pi_1(\Sigma(p, q, npq + 1))$. According to D. Johnson [2], we can regard the eigenvalues of generators of the fundamental group as a parameter of conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations. Johnson derived this description through $(1/n)$-surgery along the (p, q)-torus knot. We first review conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations for the (p, q)-torus knot exterior E_K. Here we choose the presentation $\langle x, y \mid x^p = y^q\rangle$ for $\pi_1(E_K)$.

Proposition 3.1 ([2, 5]). *Let ρ be an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(E_K)$. Then there exists the pair (a, b) of integers such that*

(i) $0 < a < p$, $0 < b < q$ *and* $a \equiv b \pmod 2$;
(ii) *the eigenvalues of $\rho(x)$ are given by* $e^{\pm a\pi\sqrt{-1}/p}$;
(iii) *the eigenvalues of $\rho(y)$ are given by* $e^{\pm b\pi\sqrt{-1}/q}$.

Conversely, each pair (a, b) satisfying the condition (i) *corresponds to the conjugacy class of an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation ρ satisfying the conditions* (ii) *and* (iii).

These conditions are derived from the requirement that the center of $\pi_1(E_K)$ should be sent into the center of $\mathrm{SL}_2(\mathbb{C})$. As a consequence of Proposition 3.1, the image $\rho(z)$ of the central element z is given by $(-I)^a(= (-I)^b)$.

We can deduce the following correspondence between triples of integers and conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations for $\Sigma(p, q, npq + 1)$ by $(1/n)$-surgery along the (p,q)-torus knot. Note that the eigenvalues for a meridian can move in the conjugacy class of any irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of a torus knot group.

Proposition 3.2 ([2], Introduction in [3]). *Suppose that ρ is an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(\Sigma(p,q,npq+1))$. Then the conjugacy class of ρ corresponds to the triple (a,b,c) of integers such that*

(i) $0<a<p$, $0<b<q$ *and* $a \equiv b \mod 2$*;*
(ii) $0<c<r=|npq+1|$ *and* $c \equiv na \pmod 2$*;*
(iii) *the eigenvalues of $\rho(x)$ are given by* $e^{\pm a\pi\sqrt{-1}/p}$*;*
(iv) *the eigenvalues of $\rho(y)$ are given by* $e^{\pm b\pi\sqrt{-1}/q}$*;*
(v) *the eigenvalues of $\rho(m)$ are given by* $e^{\pm c\pi\sqrt{-1}/r}$

where m denotes the meridian of the (p,q)-torus knot, given by the equality that $m = x^{-u}y^{v}$ $(pv-qu=1)$ in $\pi_1(E_K)$.

Conversely a triple (a,b,c) satisfying the conditions (i) *and* (ii) *corresponds to the conjugacy class of an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation ρ satisfying the conditions* (iii), (iv) *and* (v).

We have chosen the pair of $x^{-u}y^{v}$ $(pv-qu=1)$ and $\ell = m^{pq}x^{-p}$ as a peripheral system. The conditions on m in Proposition 3.2 are derived from the equality that $m\ell^{n}=1$ in $\pi_1(\Sigma(p,q,npq+1))$. We can also derive the equality that $\rho(m)^r = (-I)^{an}$.

We will see that the twisted chain complex $C_*(E_K;V_{2N})$ and $C_*(\Sigma(p,q,npq+1);V_{2N})$ are acyclic for all N under the condition that ρ sends z to $-I$ in Subsection 3.3. Precisely, the twisted chain complex $C_*(E_K;V_{2N})$ are acyclic for all N if and only if ρ sends z to $-I$. The condition that $\rho(z)=-I$ also gives a sufficient condition for all $C_*(\Sigma(p,q,npq+1);V_{2N})$ to be acyclic.

3.2. **Asymptotics of Reidemeister torsion for Brieskorn manifolds.** We observe the higher dimensional Reidemeister torsion for $\Sigma(p,q,npq+1)$. First we describe an explicit form of $(2N)$-th higher dimensional Reidemeister torsion for all N. Next we will discuss the asymptotic behavior of the sequence given by $\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|$.

Theorem 3.3. *Suppose that the conjugacy class of ρ for $\pi_1(\Sigma(p,q,npq+1))$ corresponds to a triple (a,b,c) such that $a\equiv b\equiv 1 \mod 2$. Then the twisted chain complex $C_*(\Sigma(p,q,npq+1);V_{2N})$ is acyclic and the higher dimensional Reidemeister torsion is expressed as*

$$\begin{aligned}
&\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N}) \\
(1)\qquad &= \frac{2^{2N}}{\prod_{k=1}^{N}\left\{4\sin^2\frac{(2k-1)a\pi}{2p}\cdot 4\sin^2\frac{(2k-1)b\pi}{2q}\cdot 4\sin^2\frac{(2k-1)(cpq-r)\pi}{2r}\right\}}
\end{aligned}$$

for all $N\geq 1$.

Remark 3.4. The acyclicity condition mentioned in Section 1 is that $a\equiv b\equiv 1 \pmod 2$.

The numerator of (1) is given by the Reidemeister torsion for the annulus in E_K. In the denominator of (1), the factors $4\sin((2k-1)a\pi/(2p))$ and $4\sin((2k-1)b\pi/(2q))$ come from the Reidemeister torsions for solid tori in E_K and the factors $4\sin((2k-1)(cpq-r)\pi/(2r))$ is given by the Reidemeister torsion for the solid torus glued to E_K. Theorem 3.3 follows from the following Lemma 3.5, which will be shown in Subsection 3.3.

Lemma 3.5. *Under the assumption of Theorem 3.3, the Reidemeister torsion for the Brieskorn manifold* $\Sigma(p,q,npq+1)$ *is expressed as*

$$\begin{aligned}&\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})\\&=\mathrm{Tor}(S^1_x;\rho_{2N})\cdot\mathrm{Tor}(S^1_y;\rho_{2N})\cdot\mathrm{Tor}(S^1_\ell;\rho_{2N})\cdot\mathrm{Tor}(S^1_z;\rho_{2N})^{-1}\end{aligned}$$

where each suffix of S^1 *denotes the homotopy class in* $\pi_1(\Sigma(p,q,npq+1))$.

It follows from Theorem 3.3 that the logarithm of $|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|$ is a linear combination of the logarithms of the Reidemeister torsions for the circles. The author have shown in [11] that the asymptotic behavior of the higher dimensional Reidemeister torsion for S^1.

Proposition 3.6 (Proposition 3.8 in [11]). *Let* ρ *be an* $\mathrm{SL}_2(\mathbb{C})$*-representation of* $\pi_1(S^1)=\langle\gamma\rangle$. *If* $\rho(\gamma)$ *has the order of* $2d$, *then we have the following limits:*

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(S^1;\rho_{2N})|}{(2N)^2}=0, \tag{2}$$

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(S^1;\rho_{2N})|}{2N}=-\frac{1}{d}\log 2. \tag{3}$$

By Lemma 3.5 and Proposition 3.6, we can deduce the asymptotics of the higher dimensional Reidemeister torsion for $\Sigma(p,q,npq+1)$ as follows.

Theorem 3.7. *Suppose that an irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *corresponds to a triple* (a,b,c) *such that* $a\equiv b\equiv 1 \bmod 2$. *We have the following limits which express the order of growth for the sequence given by* $\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|$.

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|}{(2N)^2}=0, \tag{4}$$

$$\lim_{N\to\infty}\frac{\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|}{2N}=\left(1-\frac{1}{p'}-\frac{1}{q'}-\frac{1}{r'}\right)\log 2 \tag{5}$$

where $p'=p/(p,a)$, $q'=q/(q,b)$ *and* $r'=r/(r,c)$.

We have only finitely many conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations for every Brieskorn manifold $\Sigma(p,q,npq+1)$. Hence we have finitely many possibilities of the limits for the leading coefficient of $\log|\mathrm{Tor}(\Sigma(p,q,npq+1);\rho_{2N})|$.

Remark 3.8. We can regard every Brieskorn manifold $\Sigma(p,q,npq+1)$ as a Seifert manifold. The limits (5) in Theorem 3.7 are less than or equal to $-\chi\log 2$ where $\chi(=1-1/p-1/q-1/r)$ is the Euler characteristic of the base orbifold of the Seifert manifold.

Corollary 3.9. *For every Brieskorn manifold* $\Sigma(p,q,npq+1)$, *there exists an acyclic irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *which gives the maximal value* $-\chi\log 2$ *in the set of limits in Eq.* (5).

Proof of Corollary 3.9. It is sufficient to find a triple (a,b,c) satisfies

- $0<a<p$, $0<b<q$ and $0<c<r$;
- $a\equiv b\equiv 1$ and $c\equiv na \pmod 2$;
- $(a,p)=(b,q)=(c,r)=1$.

It is easy to see that the triple $(1,1,|n|)$ satisfies the above conditions. □

3.3. **Reidemeister torsion of $\Sigma(p,q,npq+1)$ by Mayer–Vietoris arguments.** We compute the Reidemeister torsion for $\Sigma(p,q,npq+1)$ by Mayer–Vietoris arguments. Our computation is based on the decomposition of $\Sigma(p,q,npq+1)$ as

$$\begin{aligned}\Sigma(p,q,npq+1) &= E_K \cup_{\partial E_K} D^2 \times S^1_\ell \\ &= (D^2 \times S^1_x \cup_{S^1_z \times [-1,1]} S^1_y \times D^2) \cup_{\partial E_K} D^2 \times S^1_\ell.\end{aligned}$$

The Multiplicativity property of the Reidemeister torsion allows us to use a cut and paste method for decomposition of CW–complexes.

Lemma 3.10 (Multiplicativity property). *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be a short exact sequence of based acyclic chain complexes. Suppose that each chain complex consists of vector spaces and the basis of C_* is given by the bases of C'_* and C''_*. Then we have the following equality of the Reidemeister torsions:*

$$\mathrm{Tor}(C_*) = \pm\mathrm{Tor}(C'_*)\mathrm{Tor}(C''_*).$$

For details on the Multiplicativity property, we refer to Turaev's book [9] and Milnor's survey article [7].

We use this property for each decomposition of $E_K = D^2 \times S^1_x \cup_{S^1_z \times [-1,1]} S^1 \times D^2$ and $\Sigma(p,q,npq+1) = E_K \cup_{\partial E_K} D^2 \times S^1_\ell$. To apply Lemma 3.10 (Multiplicativity property), we need to check that every twisted chain complex in the decomposition is acyclic.

We first observe the decomposition of the (p,q)-torus knot exterior E_K.

Proposition 3.11 (Proposition 3.1 in [10]). *Let ρ be an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(E_K)$. The twisted chain complex $C_*(E_K;V_{2N})$ is acyclic for all $N \geq 1$ if and only if the pair (a,b) corresponding to the conjugacy class of ρ satisfies that $a \equiv b \equiv 1 \pmod 2$.*

Remark 3.12. For any irreducible $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(E_K)$, the condition that $a \equiv b \equiv 1 \pmod 2$ is equivalent to $\rho(z) = -I$ since $\rho(z) = (-I)^a$.

Under the condition which requires that z is sent to $-I$, we can also see the acyclicity for every twisted chain complex in the decomposition of $E_K = D^2 \times S^1_x \cup_{S^1_z \times [-1,1]} S^1_y \times D^2$. The twisted chain complexes $C_*(D^2 \times S^1_x;V_{2N})$, $C_*(S^1_y \times D^2;V_{2N})$ and $C_*(S^1_z \times [-1,1];V_{2N})$ are defined by the restrictions of an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(E_K)$.

Proposition 3.13. *Let ρ be an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(E_K)$. All of the twisted chain complexes $C_*(D^2 \times S^1_x;V_{2N})$, $C_*(S^1_y \times D^2;V_{2N})$ and $C_*(S^1_z \times [-1,1];V_{2N})$ are acyclic if and only if the image $\rho(z)$ is equal to $-I$.*

Proof. The Mayer–Vietoris sequence of twisted homology groups is expressed as

$$\cdots \to H_i(S^1_z \times [-1,1]) \to H_i(D^2 \times S^1_x) \oplus H_i(S^1_y \times D^2) \to H_i(E_K) \to \cdots$$

where each coefficient is V_{2N}.

We assume that $\rho(z)$ is $-I$. It follows from the Mayer-Vietoris sequence and Proposition 3.11 that the twisted homology group $H_*(S^1_z \times [-1,1];V_{2N}) \simeq H_*(S^1_z;V_{2N})$ is isomorphic to $H_*(D^2 \times S^1_x;V_{2N}) \oplus H_*(S^1_y \times D^2;V_{2N})$. By Corollary 2.5, we can see that $H_*(S^1_z;V_{2N}) = 0$. Therefore all of the twisted chain complexes $C_*(D^2 \times S^1_x;V_{2N})$, $C_*(S^1_y \times D^2;V_{2N})$ and $C_*(S^1_z \times [-1,1];V_{2N})$ are acyclic.

Next we assume that all of twisted homology groups for $D^2 \times S^1_x$, $S^1_y \times D^2$ and $S^1_z \times [-1,1]$ are trivial. Then the twisted homology group $H_*(E_K;V_{2N})$ also vanishes from the Mayer-Vietoris sequence. By Proposition 3.11, we can conclude that $\rho(z) = -I$. $\square$

Now we are in a position to apply Lemma 3.10 (Multiplicativity property) to the short exact sequence:

$$0 \to C_*(S^1_z \times [-1,1]; V_{2N}) \to C_*(D^2 \times S^1_x; V_{2N}) \oplus C_*(S^1_y \times D^2; V_{2N}) \to C_*(E_K; V_{2N}) \to 0.$$

Proposition 3.14. *Suppose that an irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *of* $\pi_1(E_K)$ *sends* z *to* $-I$*. Then the higher Reidemeister torsion* $\mathrm{Tor}(E_K; \rho_{2N})$ *is expressed as*

$$\mathrm{Tor}(E_K;\rho_{2N}) = \mathrm{Tor}(D^2 \times S^1_x; \rho_{2N}) \cdot \mathrm{Tor}(S^1_y \times D^2; \rho_{2N}) \cdot \mathrm{Tor}(S^1_z \times [-1,1]; \rho_{2N})^{-1} \tag{6}$$

$$= \frac{2^{2N}}{\prod_{k=1}^{N} 4 \sin \frac{(2k-1)a\pi}{2p} \cdot 4 \sin \frac{(2k-1)b\pi}{2q}} \tag{7}$$

where a *and* b *are integers whose pair* (a,b) *corresponds to the conjugacy class of* ρ*.*

Proof of Proposition 3.14. Eq. (6) follows from Lemma 3.10. By Corollary 2.5, each of the Reidemeister torsions in Eq. (6) is expressed as follows:

$$\begin{aligned}
\mathrm{Tor}(S^1_z \times [-1,1]; \rho_{2N}) &= \{\det(\rho_{2N}(z)^{-1} - I_{2N})\}^{-1} \\
&= (-2)^{-2N}, \\
\mathrm{Tor}(D^2 \times S^1_x; \rho_{2N}) &= \mathrm{Tor}(S^1_x; \rho_{2N}) \\
&= \left\{ \prod_{k=1}^{N} (e^{(2k-1)a\pi\sqrt{-1}/p} - 1)(e^{-(2k-1)a\pi\sqrt{-1}/p} - 1) \right\}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tor}(S^1_y \times D^2; \rho_{2N}) &= \mathrm{Tor}(S^1_y; \rho_{2N}) \\
&= \left\{ \prod_{k=1}^{N} (e^{(2k-1)b\pi\sqrt{-1}/q} - 1)(e^{-(2k-1)b\pi\sqrt{-1}/q} - 1) \right\}^{-1}.
\end{aligned}$$

We complete the proof by substituting the above computations into Eq. (6). □

Next we apply Lemma 3.10 (Multiplicativity property) to the short exact sequence for the decomposition that $\Sigma(p,q,npq+1) = E_K \cup_{\partial E_K} D^2 \times S^1_\ell$. As seen in the case that of E_K, we need to check the acyclicity of twisted chain complexes. We regard $\mathrm{SL}_2(\mathbb{C})$-representations for the resulting manifold $\Sigma(p,q,npq+1)$ as the extensions of irreducible $\mathrm{SL}_2(\mathbb{C})$-ones ρ of $\pi_1(E_K)$ such that $\rho(m\ell^n) = I$.

Lemma 3.15. *Let* ρ *be an irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representation of* $\pi_1(\Sigma(p,q,npq+1))$*. If* ρ *sends* z *to* $-I$*, then the order of* $\rho(\ell)$ *is even. In particular, under the condition that* $\rho(z) = -I$*, the twisted chain complex* $C_*(D^2 \times S^1_\ell; V_{2N})$ *is acyclic for all* $N \geq 1$*.*

Proof. Set (a,b,c) as the triple of integers corresponding to the conjugacy class of ρ. By Proposition 3.2, we can see that $\rho(\ell)^r = (-I)^a$ as follows:

$$\rho(\ell)^r = \rho(m)^{pqr} \rho(x^{-p})^r = (-I)^{pqc}(-I)^{-a(npq+1)} = (-I)^{-a}.$$

The condition that $\rho(z) = -I$ is equivalent to $a \equiv b \equiv 1 \pmod 2$. This implies that $\rho(\ell)$ has the order of even degree if $\rho(z) = -I$.

Therefore it follows from Corollary 2.5 that $H_*(D^2 \times S^1_\ell; V_{2N}) \simeq H_*(S^1_\ell; V_{2N})$ vanishes. □

By applying Proposition 3.11 and Lemma 3.15 to the Mayer–Vietoris sequence:

$$\cdots \to H_i(\partial E_K) \to H_i(E_K) \oplus H_i(D^2 \times S^1_\ell) \to H_i(\Sigma(p,q,npq+1)) \to \cdots \tag{8}$$

with the coefficient V_{2N}, we can obtain the following acyclicity of the twisted chain complexes under the condition that $\rho(z) = -I$.

Proposition 3.16. *If an irreducible* $\mathrm{SL}_2(\mathbb{C})$*-representation* ρ *of* $\pi_1(\Sigma(p,q,npq+1))$ *satisfies that* $\rho(z) = -I$*, then all of* $C_*(\Sigma(p,q,npq+1); V_{2N})$*,* $C_*(E_K; V_{2N})$*,* $C_*(D^2 \times S^1_\ell; V_{2N})$ *and* $C_*(\partial E_K; V_{2N})$ *are acyclic for all* $N \geq 1$*.*

Proof. It follows from Proposition 3.11 and Lemma 3.15 that $C_*(E_K; V_{2N})$ and $C_*(D^2 \times S^1_\ell; V_{2N})$ are acyclic for all $N \geq 1$. Since for any $N \geq 1$, $\rho_{2N}(\ell)$ does not have the eigenvalue 1, we can show that $C_*(\partial E_K; V_{2N})$ is acyclic for all N by direct calculation. Hence the acyclicity of $C_*(\Sigma(p,q,npq+1))$ follows from the Mayer–Vietoris sequence (8). □

Now we can apply Lemma 3.10 to the decomposition $\Sigma(p,q,npq+1) = E_K \cup D^2 \times S^1_\ell$ under the condition that $\rho(z) = -I$.

Proof of Lemma 3.5. Suppose that an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation ρ sends z to $-I$. We have seen the acyclicity of $C_*(\Sigma(p,q,npq+1); V_{2N})$ in Proposition 3.16. By applying Lemma 3.10 to the short exact sequence:

$$0 \to C_*(\partial E_K; V_{2N}) \to C_*(E_K; V_{2N}) \oplus C_*(D^2 \times S^1_\ell; V_{2N}) \to C_*(\Sigma(p,q,npq+1); V_{2N}) \to 0,$$

we have the following equality of the Reidemeister torsions:

$$\mathrm{Tor}(\Sigma(p,q,npq+1); \rho_{2N}) = \pm \mathrm{Tor}(E_K; \rho_{2N}) \cdot \mathrm{Tor}(D^2 \times S^1_\ell; \rho_{2N}) \cdot \mathrm{Tor}(\partial E_K; \rho(2N))^{-1} \tag{9}$$

We can see that $\mathrm{Tor}(\partial E_K; \rho_{2N}) = 1$ by definition. Together with Eq. (6) in Proposition 3.14, we obtain the equality of the Reidemeister torsions in Lemma 3.5. □

Proof of Theorem 3.3. To compute $\mathrm{Tor}(D^2 \times S^1_\ell; \rho_{2N})$, we need to consider the eigenvalues of $\rho(\ell)$. The relation $\ell = m^{pq}x^{-p} (= m^{pq}z^{-1})$ implies that the eigenvalues of $\rho(\ell)$ are $e^{\pm(cpq-r)\pi\sqrt{-1}/r}$ by the assumption that the eigenvalues of $\rho(m)$ are $e^{\pm c\pi\sqrt{-1}/r}$ where $r = |npq+1|$. Hence the Reidemeister torsion for $D^2 \times S^1_\ell$ is expressed as

$$\mathrm{Tor}(D^2 \times S^1_\ell; \rho_{2N}) = \left\{ \prod_{k=1}^{N} 4 \sin^2 \frac{(cpq-r)\pi}{2r} \right\}^{-1}. \tag{10}$$

Substituting Proposition 3.14 and Eq. (10) into Lemma 3.5, we obtain the desired equality. □

Remark 3.17. Since the coefficient V_{2N} is an even dimensional vector space, we do not need the sign $\pm$ in Eq. (9) in fact.

4. On some Seifert surgeries along the figure eight knot with SnapPy

We also touch a relation to the result of P. Menal–Ferrer and J. Porti [6]. They have shown the relation between the hyperbolic volume of a hyperbolic 3-manifold and the leading coefficient of its higher dimensional Reidemeister torsion. It is expressed as

$$\lim_{N \to \infty} \frac{\log |\mathrm{Tor}(M; \rho_{2N})|}{(2N)^2} = -\frac{\mathrm{Vol}(M)}{4\pi} \tag{11}$$

where ρ_{2N} is induced by the holonomy representation corresponding to the complete hyperbolic structure of M.

We can consider the volume of an $\mathrm{SL}_2(\mathbb{C})$-representation. The volume changes continuously when we move $\mathrm{SL}_2(\mathbb{C})$-representations. In the case that M is the interior of the figure eight knot exterior, the volume is expected to be zero when we move the holonomy representation to $\mathrm{SL}_2(\mathbb{C})$-representations corresponding to the slopes of Seifert surgeries. Here Seifert surgery means that the resulting manifold turns into a Seifert manifold. We denote by 4_1 the figure eight knot. We can work on numerical experiments with SnapPy [1] which is a program for studying the topology and geometry of 3-manifolds. SnapPy calculates the hyperbolic volume of $S^3 \setminus 4_1$ and the resulting manifold by (-1)-surgery.

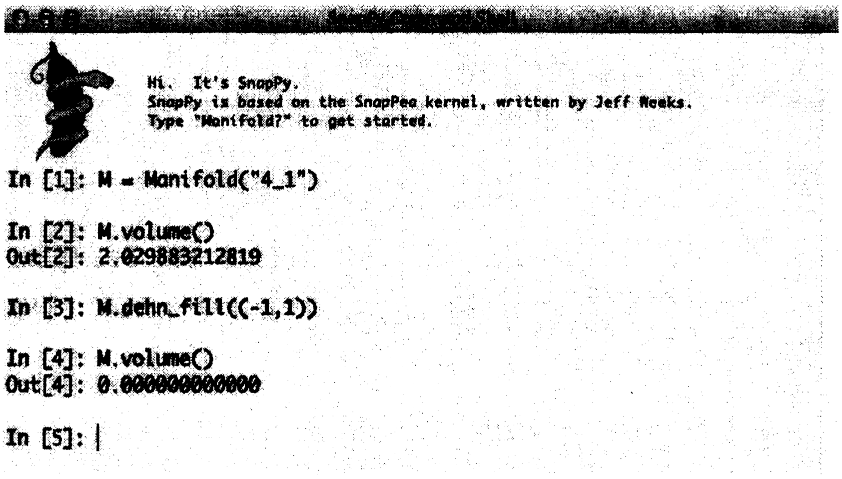

FIGURE 2. Screenshot of SnapPy

It is known that (-1)-surgery along 4_1 yields the Seifert manifold obtained by 1-surgery along the trefoil knot (we refer to [8]). Since the trefoil knot is the $(2,3)$-torus knot, the resulting manifold is the Brieskorn manifold $\Sigma(2,3,7)$.

Let ρ be an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of the figure eight knot group such that $\rho(\mu\lambda^{-1}) = I$ where μ is a meridian and λ is a longitude. It follows from our computations in Section 3 that the growth order of $\log|\mathrm{Tor}(S^3_{4_1}(-1);\rho_{2N})|$ is $2N$ and the coefficient $\log|\mathrm{Tor}(S^3_{4_1}(-1);\rho_{2N})|/(2N)$ converges $-\chi\log 2$ where $S^3_{4_1}(-1)$ is the resulting manifold by (-1)-surgery along 4_1.

It is known that the conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(S^3\setminus 4_1)$ form a set which we can equip with the structure of an affine variety. The Reidemeister torsion and the volume of a representation have the invariance under the conjugation of representations. Eq. (11) can be regard as an equality of functions on a neighbourhood of the conjugacy class of ρ. We can rephrase the above observation as follows.

The leading coefficient of $\log|\mathrm{Tor}(S^3_{4_1}(-1);\rho_{2N})|$ vanishes at the conjugacy class of ρ and the second coefficient converges to $-\chi\log 2$ where χ is the Euler characteristic of the base orbifold for the resulting Seifert manifold.

ACKNOWLEDGMENT

The author had started the study on the asymptotics of the higher dimensional Reidemeister torsion for torus knot exteriors and Seifert manifolds after the workshop "RIMS Seminar: Representation spaces, twisted topological invariants and geometric structures of 3-manifolds" at Hakone. These studies were motivated by the work by Pere Menal–Ferrer and Joan Porti and the presentation by Joan Porti in the workshop. The author gratefully acknowledges the helpful suggestions of Joan Porti. The author also would like to express his thanks to the organizers, Takayuki Morifuji, Yasushi Yamashita and Teruaki Kitano for inviting him to the workshop.

References

[1] M. Culler, N. Dunfield and et al., *SnapPy*, http://www.math.uic.edu/t3m/SnapPy/index.html

[2] D. Johnson, *A geometric form of Casson's invariant and its connection to Reidemeister torsion*, unpublished lecture notes.

[3] T. Kitano, *Reidemeister torsion of Seifert Fibered Spaces for* $SL(2;\mathbb{C})$*-Representations*, Tokyo J. Math, **17** (1994), 59–75.

[4] T. Kitano, *Reidemeister torsion of Seifert fibered spaces for* $\mathrm{SL}(n;\mathbb{C})$*-representations*, Kobe J. Math., **13** (1996), 133–144.

[5] T. Kitano and T. Morifuji, *Twisted Alexander polynomials for irreducible* $\mathrm{SL}(2;\mathbb{C})$*-representations of torus knots*, Annali della Scuola Normale Superiore di Pisa Classe di Scienze **11** (2012), 395–406.

[6] P. Menal–Ferrer and J. Porti, *Higher dimensional Reidemeister torsion invariants for cusped hyperbolic* 3*-manifolds*, arXiv:1110.3718.

[7] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc., **72** (1966), 358–426.

[8] K. Motegi, *An Experimental Study of Seifert Fibered Dehn Surgery via SnapPea*, Interdisciplinary Information Sciences, **9** (2003), 95–125.

[9] V. Turaev, *Introduction to Combinatorial Torsions*, Lectures in Mathematics (2001) Birkhäuser.

[10] Y. Yamaguchi, *Higher even dimensional Reidemeister torsion for torus knot exteriors*, arXiv:1208.4452.

[11] Y. Yamaguchi, *A surgery formula for the asymptotics of the higher dimensional Reidemeister torsion and Seifert fibered spaces*, arXiv:1210.8049.

Department of Mathematics, Tokyo Institute of Technology
E-mail address: shouji@math.titech.ac.jp